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On the symmetry of 9- and 10-hedra

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The symmetry point groups for all combinatorially non-isomorphic 9- and 10-hedra (2606 and 32300, respectively) are contributed in the paper for the first time. The most symmetrical polyhedra of 3 to 32 automorphism group orders (50 and 187, respectively) are drawn in the Schlegel projections and characterized by the facet symbols and symmetry point groups.

1. Introduction

The symmetry of 4- to 11-hedra was discussed in our papers (Voytekhovsky, 1999, 2000, 2001*a*) with some disagreements between the previous data being eliminated. But only simple (three facets meet at each vertex) 9-, 10- and 11-hedra were studied there. Here, we contribute the symmetry point groups for all not simple non-isomorphic 9- and 10-hedra for the first time. For the sake of completeness, the symmetry statistics for the simple 9- and 10-hedra are also included in the tables.

2. Generation and characterization of polyhedra

We generate the polyhedra as their Schlegel projections. This approach is obviously justified by the Steinitz theorem (*every 3-connected planar graph can be realized as a 3-polyhedron*) and the Mani theorem (*every combinatorial automorphism of a 3-polyhedron is affinely realizable*). That is, there exists to each Schlegel diagram a 3-space realization of a polyhedron such that its edge graph is isomorphic to the Schlegel diagram while its symmetry point group is isomorphic to the automorphism group of the Schlegel diagram.

The polyhedra were generated by the Fedorov (1893) recurrence algorithm briefly described by Engel (1994) and Voytekhovsky (2001*b*). As the simple 9- and 10-hedra were previously found, we used them to generate not simple polyhedra by the reduction operation ω which is known to reduce any edge v_1–v_2 (joining two adjacent vertices v_1 and v_2) if all facets containing v_1 but not v_2 have no common vertex with any facet containing v_2 but not v_1 (Fedorov, 1893, p. 281). Applying ω step by step in all possible ways, we reduced the number of vertices from 14 to 7 at 9-hedra and from 16 to 7 at 10-hedra. The generated polyhedra were compared in the Schlegel projections. Afterwards, the combinatorially non-isomorphic polyhedra were characterized by their facet symbols and symmetry point groups. A facet symbol $[n_3 n_4 \ldots n_{max}]$ shows the sequence of numbers of triangular, quadrilateral *etc.* facets at a polyhedron.

a.g.o.	s.p.g.	Vertices 7	8	9	10	11	12	13	14	Total
1	1	2	48	237	533	662	449	164	16	2111
2	m	4	17	48	71	87	74	46	18	365
	2		5	8	22	10	25	3	7	80
3	3					1				1
4	mm2	2	1		7	5	10	4	5	34
6	3m		2			2			1	5
8	4mm			2				2		4
12	-6m2		1			1			2	4
16	8mm			1						1
28	-14m2								1	1
Total		8	74	296	633	768	558	219	50	2606

Figure 1
The automorphism group order (a.g.o.) and symmetry point group (s.p.g.) statistics of 9-hedra.

3. Results and discussion

The automorphism group order and symmetry point group statistics of 9- and 10-hedra are in given Figs. 1 and 2. The most symmetrical

a.g.o.	s.p.g.	Vertices 7	8	9	10	11	12	13	14	15	16	Total
1	1		44	533	2401	5790	8331	7491	4052	1235	137	30014
2	m		19	71	145	273	347	341	275	143	51	1665
	2	1	8	22	56	58	109	55	87	13	18	427
	-1				1		3		3			7
3	3				3			4			1	8
4	mm2	1	2	7	14	13	17	11	19	13	11	108
	2/m		1		4		6		4		2	17
	222				1		3		1			5
	-4						1					1
6	3m	2			9			14			6	31
8	-42m		1				1				1	3
	4mm						2				1	3
	mmm								1		2	3
16	4/mmm						1					1
	-82m		1								1	2
18	9m				1							1
20	-10m2	1										1
	-5m						1					1
24	-43m										1	1
32	8/mmm										1	1
Total		5	76	633	2635	6134	8822	7916	4442	1404	233	32300

Figure 2
The automorphism group order (a.g.o.) and symmetry point group (s.p.g.) statistics of 10-hedra.

polyhedra with the automorphism group orders being not less than 3 are given in the Schlegel projections in Figs. 3 and 4. Their facet symbols (given in square brackets) and symmetry point groups (shown by bold symbols) are as follows.

9-hedra: [81] **mm2**: 1, 2. [63] **3m**: 3, 4; **$\bar{6}$m2**: 5. [8001] **mm2**: 6. [45] **4mm**: 7, 8. [800001] **8mm**: 9. [27] **mm2**: 10, 11. [432] **mm2**: 12. [4401] **mm2**: 13–15. [620001] **mm2**: 16. [09] **$\bar{6}$m2**: 17. [2601] **mm2**: 18. [333] **3**: 19; **3m**: 20, 21. [414] **mm2**: 22. [4221] **mm2**: 23. [440001] **mm2**: 24, 25. [072] **mm2**: 26, 27. [0801] **mm2**: 28. [234] **mm2**: 29. [260001] **mm2**: 30. [4041] **mm2**: 31–33. [4122] **mm2**: 34. [4203] **mm2**: 35. [054] **4mm**: 36. [0621] **mm2**: 37. [2241] **mm2**: 38. [2322] **mm2**: 39. [25002] **mm2**: 40. [404001] **4mm**: 41. [036] **$\bar{6}$m2**: 42. [0441] **mm2**: 43. [0522] **mm2**: 44. [0603] **$\bar{6}$m2**: 45. [07002] **$\overline{14}$m2**: 46. [2304] **mm2**: 47. [250002] **mm2**: 48. [3033] **3m**: 49. [402201] **mm2**: 50.

10-hedra: [10] **mm2**: 1; **3m**: 2, 3; **$\overline{10}$m2**: 4. [82] **mm2**: 5, 6; **2/m**: 7; **$\bar{4}$2m**: 8; **$\bar{8}$2m**: 9. [64] **mm2**: 10–14. [802] **mm2**: 15, 16. [46] **3**: 17; **222**: 18; **mm2**: 19–26; **2/m**: 27–29; **3m**: 30–33. [622] **mm2**: 34–38; **2/m**: 39. [6301] **3**: 40; **mm2**: 41; **3m**: 42, 43. [703] **3**: 44; **3m**: 45–47. [9000001] **9m**: 48. [28] **mm2**: 49–52. [442] **mm2**: 53–59. [6202] **mm2**: 60, 61. [0,10] **222**: 62; **4/mmm**: 63; **$\bar{5}$m**: 64. [262] **mm2**: 65–71; **2/m**: 72. [2701] **mm2**: 73. [424] **222**: 74; **mm2**: 75–77; **$\bar{4}$**: 78; **2/m**: 79, 80; **4mm**: 81. [4402] **222**: 82; **mm2**: 83–85; **2/m**: 86–88; **$\bar{4}$2m**: 89. [450001] **mm2**: 90; **4mm**: 91. [6022] **mm2**: 92, 93. [082] **mm2**: 94. [0901] **3m**: 95. [163] **3m**: 96–98. [244] **mm2**: 99–101. [2602] **mm2**: 102, 103. [3331] **3**: 104; **3m**: 105, 106. [3600001] **3m**: 107. [406] **3**: 108; **3m**: 109, 110. [4222] **mm2**: 111–113. [4303] **3**: 114; **3m**: 115–117. [6004] **3**: 118; **mm2**: 119; **3m**: 120. [60202] **mm2**: 121. [6030001] **3m**: 122. [064] **222**: 123; **mm2**: 124, 125; **2/m**: 126. [0802] **mm2**: 127. [226] **mm2**: 128–130; **2/m**: 131. [2341] **mm2**: 132. [2422] **mm2**: 133, 134. [26002] **2/m**: 135. [260101] **mm2**: 136. [4042] **mm2**: 137; **2/m**: 138; **mmm**: 139. [4123] **mm2**: 140. [414001] **mm2**: 141, 142. [4204] **mm2**: 143. [42202] **mm2**: 144–146. [60022] **mm2**: 147. [0622] **mm2**: 148, 149. [208] **mm2**: 150. [2242] **mm2**: 151, 152. [24202] **mm2**: 153–155. [260002] **mm2**: 156. [4024] **mm2**: 157, 158. [42022] **mm2**: 159. [422002] **mm2**: 160. [028] **$\bar{8}$2m**: 161. [0361] **3m**: 162. [0442] **mmm**: 163. [0523] **mm2**: 164. [0604] **$\bar{4}$2m**: 165. [06202] **mm2**: 166, 167. [080002] **8/mmm**: 168. [1333] **3m**: 169, 170. [2224] **mm2**: 171; **2/m**: 172. [22402] **2/m**: 173. [224101] **mm2**: 174. [2305] **mm2**: 175. [24022] **mm2**: 176. [242002] **mm2**: 177. [2600002] **mm2**: 178. [33013] **3m**: 179, 180. [3303001] **3**: 181. [4006] **mmm**: 182; **$\bar{4}$3m**: 183. [40222] **mm2**: 184. [40303] **3m**: 185. [410401] **4mm**: 186. [420202] **mm2**: 187.

The automorphism group order statistics agree with the data given in Duijvestijn & Federico (1981). The symmetry point group statistics are contributed here for the first time. As for 4- to 8-hedra and simple 9- to 11-hedra (Voytekhovsky, 2001*a*), the shapes of **1**, **m**, **2** and **mm2** symmetry also prevail among not simple 9- and 10-hedra. This tendency appears to be a general property of the abstract polyhedra variety. The number of polyhedra rapidly drops with growing symmetry so that trivial (of **1** symmetry point group) shapes form the overwhelming majority. The depressing fact is that it cannot be classified in the framework of the symmetry theory. We need some new approaches to do this.

4. Conclusions

Up to now, the whole variety of 4- to 10-hedra and simple 11-hedra is generated, drawn in the Schlegel projections and characterized by the facet symbols, automorphism group orders and symmetry point groups. Their overwhelming majority is found to belong to the trivial symmetry point group. The next steps are to generate and characterize in the same way all not simple 11-hedra, simple 12- and 13-hedra, and to find some methods to classify the trivial shapes of the same Euler's genera (*i.e.* class of polyhedra with the same numbers of facets, edges and vertices). They will be discussed in our next papers.

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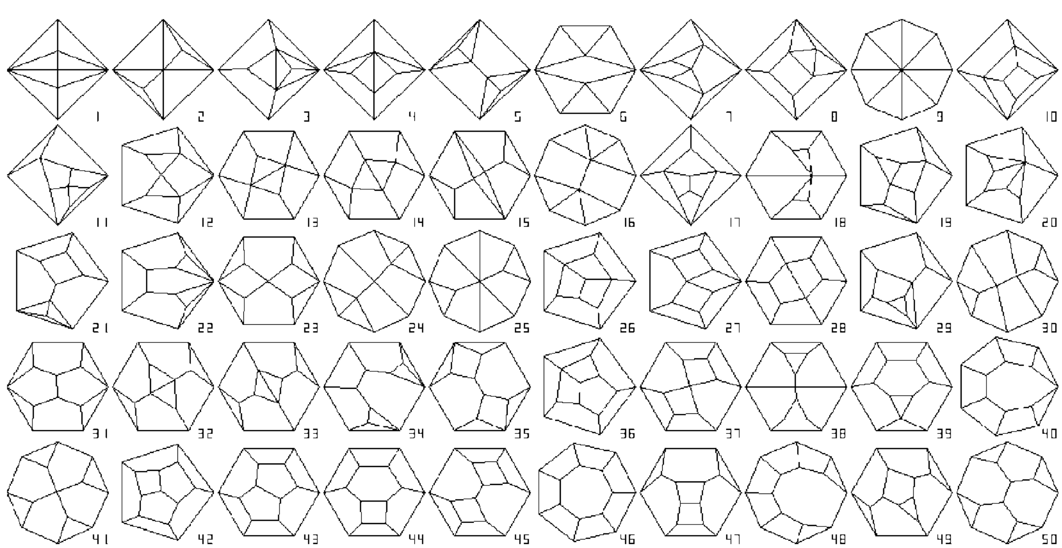

Figure 3
The most symmetrical 9-hedra in the Schlegel projections. See text for facet symbols and symmetry point groups.

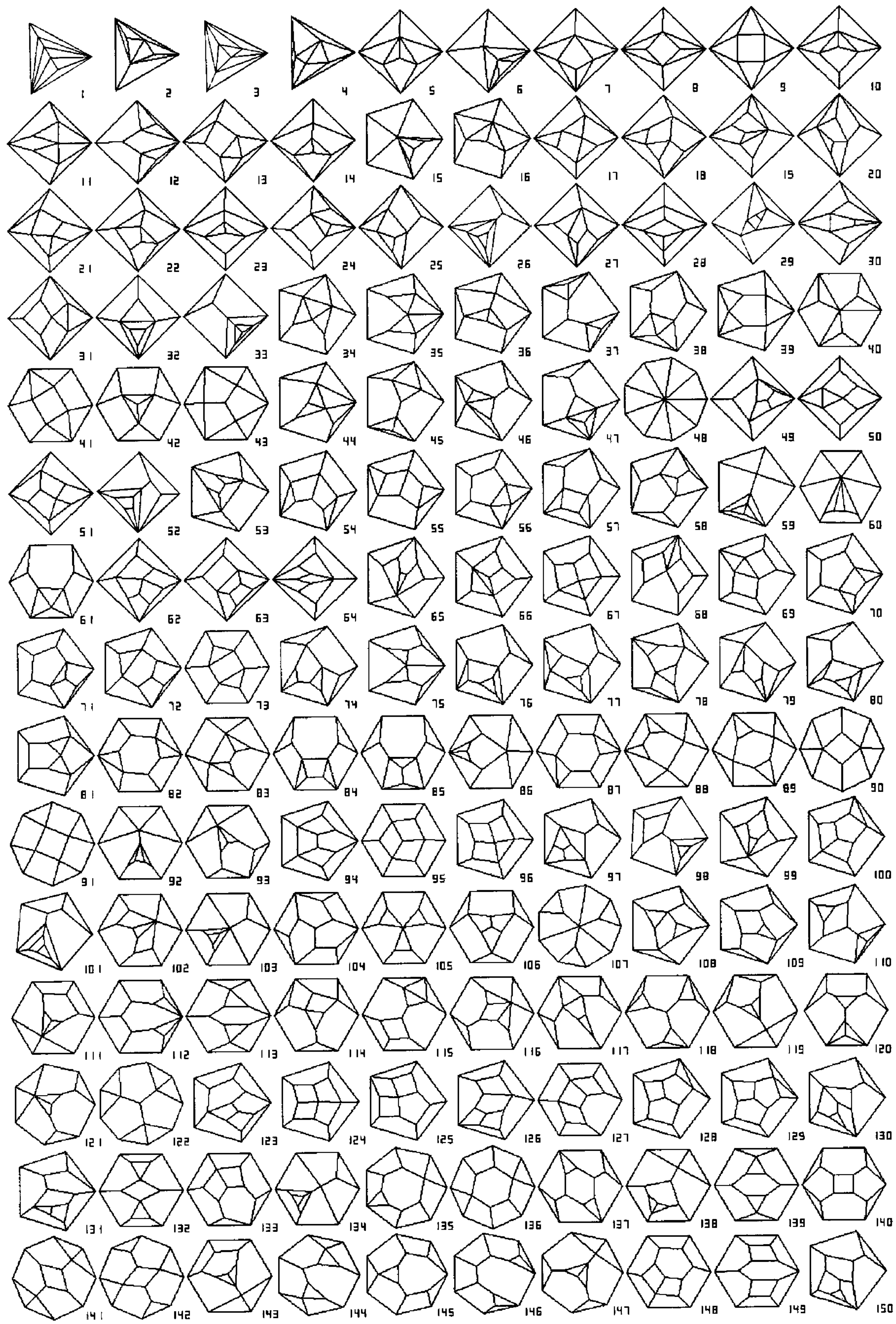

Figure 4
The most symmetrical 10-hedra in the Schlegel projections. See text for facet symbols and symmetry point groups.

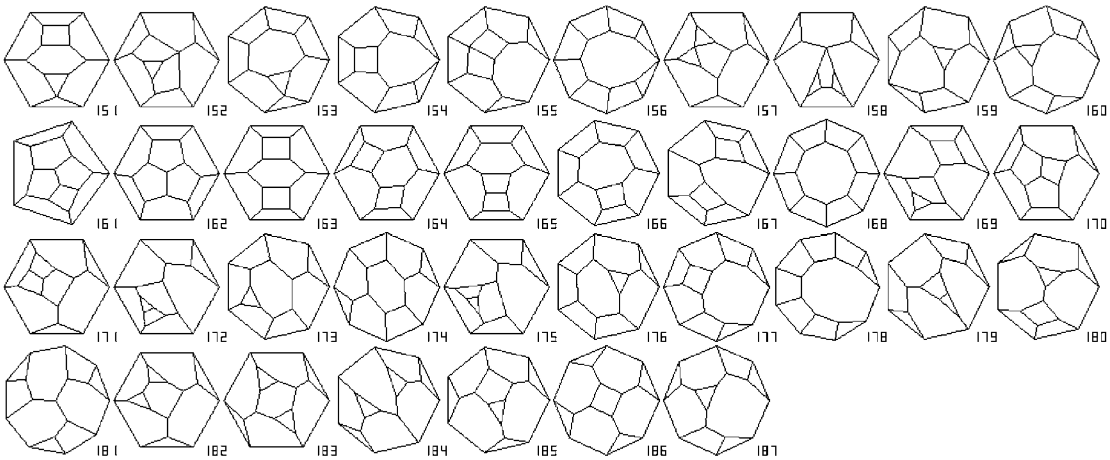

Figure 4 (continued)

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